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Probing some $\mathcal{N}=1$ AdS/CFT RG flows

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Abstract: We present the results of probing the ten-dimensional type-IIB supergravity solution corresponding to a renormalisation group flow of supersymmetric SU(N) Yang-Mills theory from pure gauge $\mathcal{N}=4$ to $\mathcal{N}=1$ with two massless adjoint flavours. The endpoint of the flow is an infrared fixed point theory, and because of this simplicity of the theory, the effective lagrangian for the probe is very well behaved, having no zeros or singularities in the tension, and a smooth potential, all of which we exhibit. Specialising to the locus of points where the potential vanishes, we also characterise a part of the Coulomb branch of the $\mathcal{N}=1$ theory. The simplicity of the gauge theory physics allows us to isolate and emphasise a key holographic feature of brane probe physics which has wider applications in the study of geometry/gauge theory duals.

Keywords: D-branes, Brane Dynamics in Gauge Theories, AdS/CFT Correspondance.

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Contents

1. Introduction

One of the examples of the AdS/CFT correspondence [1, 2, 3] is a duality between $\mathcal{N} = 4$ supersymmetric SU(N) Yang-Mills theory (at large N and strong 't Hooft coupling) and type-IIB string theory propagating on $AdS_5 \times S^5$. In this strong form (and in a number of weaker forms), is a remarkable relationship between the dynamics of gauge theory and geometry. It points to the possibility of developing many useful tools for application to the study of gauge theories relevant to very important physical phenomena, such as confinement, the phenomenology of baryonic matter at high temperature and density, etc.

One of the many avenues of investigation being explored in order to make contact with such exciting physics is the issue of deforming the correspondence by switching on relevant operators, so that it represents the renormalisation group (RG) flow from $\mathcal{N} = 4$ supersymmetric pure Yang-Mills theory in the ultraviolet (UV) to $\mathcal{N} = 2, 1$ or 0 supersymmetric Yang-Mills theories of various sorts, in the infrared (IR). A number of supergravity solutions have been constructed with such dual interpretations [4]–[21].

The supergravity solutions interpolate between $AdS_5 \times S^5$ at $r = +\infty$, the dual of the $\mathcal{N} = 4$ theory (with strong 't Hooft coupling in the UV; here r is a suitably chosen radial coordinate of AdS_5), and a solution in the interior at $r = -\infty$, dual to the new gauge theory obtained after relevant perturbation and flowing to the IR.

Recent work [22] on other gauge/geometry situations have shown that it is quite fruitful to focus on the physics which may be obtained from studying the behaviour of a single probe D-brane (of a suitable variety) in the supergravity background.[1] This has led to the discovery of significant modifications of the naive supergravity geometry, and bridges the gap between the pure supergravity technology and that of the full superstring theory which has yet to be developed for propagation in these backgrounds.[2] Those investigations have uncovered a new rationale and mechanism for removing troublesome singularities in geometries dual to gauge theories[3] The resulting "enhançon" geometry, made of smeared branes, is consistent with the dual (or accompanying) gauge theory, and often sheds new light on it.

The RG flow geometries are typically derived in the context of $\mathcal{N}=8$ supergravity in five dimensions, which is believed to be a consistent truncation of type-IIB supergravity. There are not many complete ten-dimensional "lifts" of these solutions known, and so the probe techniques (which rely on having the full geometry, since the probe brane couples to fully ten-dimensional fields) have not been applied to the study of the many RG flows which are known.

This situation is changing. Soon after the presentation in ref. [17] of the complete ten-dimensional lift of flows to certain non-conformal $\mathcal{N}=2$ theories, probe results were presented in refs. [29, 30] which revealed new information about the nature of the flow geometry and its singularities, and uncovered new information about aspects of the gauge theory as captured by the geometry. As expected from the study of such non-conformal cases in ref. [22, 24], the appearance of an enhançon modifies and clarifies a number of the purely supergravity conclusions.

There is a particular flow, found quite early in the game [10], which represents a mass deformation of the theory which flows in the IR to $\mathcal{N}=1$ supersymmetric SU(N) gauge theory (N large) with two massless adjoint "flavours". It is a fixed point theory, as known from the work of ref. [31] and clarified in refs. [10, 32]. There is a U(1) R-symmetry and an SU(2) flavour symmetry.

In the language of the gravitational dual, the fact that the IR is conformally invariant means that in both the $r=+\infty$ and the $r=-\infty$ limits the geometry is asymptotically AdS_5, while the transverse space changes from a round S^5 to a space with $\mathrm{SU}(2)\times\mathrm{U}(1)$ isometry.

Until relatively recently, the full ten-dimensional geometry of this solution was not known. However, the full ten-dimensional solution representing this flow was presented in ref. [20], and we review aspects of it in section 2.

Since this deformation of the theory flows to a conformal field theory, (we recall some of this in section 3), one does not expect to have the sort of complicated probe behaviour which was seen in the $\mathcal{N}=2$ non-conformal cases. This expectation

[1]For a collection of pedagogical studies with a focus on these techniques, see ref. [23].

[2]See also refs. [24]–[27] for extensions and further study of the approach of ref. [22]. Other recent appearances of the enhançon phenomenon may be found in refs. [28].

[3]See ref. [15] for a review of the singularities which occur in the RG flow context, and a proposal for their classification into physical and non physical within the context of supergravity.

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is borne out by the explicit computations presented in this paper. We probe the geometry with a D3-brane in section 4 and find the effective lagrangian for the motion of the probe. It is quite well behaved. Specialising to the locus of points where the potential vanishes, we see in section 5 that the Coulomb branch of moduli space is topologically $\mathbb{R}^4$, as it should be, but deformed in a way which preserves only the $\mathrm{SU}(2)_F \times \mathrm{U}(1)_R$ global symmetry. We compute and exhibit the metric on this moduli space.

In fact, the probe computation yields a four-dimensional moduli space *everywhere along the flow*, including the far ultraviolet. This is despite the fact that the supergravity solution asymptotes to $AdS_5 \times S^5$. This fits with the field theory: since the probe at radius r computes the effective physics at some cutoff set by u or r, it knows about the true vacuum structure of the field theory, which includes the fact that with the perturbation present, the moduli space has four flat directions. In effect therefore, arbitrarily far into the UV, the probe is sensitive to the IR physics, as we point out precisely how the probe computation achieves this.

We amplify this point in the discussion of section 6. There, we point out the subtle fact that extracting the results of the probe computations, interpreting them in terms of a four-dimensional field theory, emphasises the characteristic holographic nature of the D-brane probe. While supergravity probes see the features of local five or ten-dimensional physics, the brane probe physics has a four-dimensional interpretation, and therefore is sensitive to physics at very different radii, since these represent different cutoff scales. The results of the probe computation shows just how this is achieved. This is consistent with intuition about the nature of effective lagrangians, and is a key feature of how the brane captures the holographic nature of the geometry/gauge theory duality.

2. The ten-dimensional solution

The ten-dimensional solutions computed in ref. [20] describing the gravity dual of $\mathcal{N}=4$ supersymmetric $\mathrm{SU}(N)$ Yang-Mills theory, mass deformed to $\mathcal{N}=1$ in the IR may be written as:

$$ds^2_{10} = \Omega^2 ds^2_{1,4} + ds^2_5\,, \tag{2.1}$$

for the Einstein metric, where

$$ds^2_{1,4} = e^{2A(r)}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + dr^2\,, \tag{2.2}$$

and [17, 33]

$$\begin{aligned} ds^2_5 = L^2 \frac{\Omega^2}{\rho^2\cosh^2\chi}\Bigg[& d\theta^2 + \rho^6\cos^2\theta\left(\frac{\cosh\chi}{\bar{X}_2}\sigma_3^2 + \frac{\sigma_1^2+\sigma_2^2}{\bar{X}_1}\right) + \\ & + \frac{\bar{X}_2\cosh\chi\sin^2\theta}{\bar{X}_1^2}\left(d\phi + \frac{\rho^6\sinh\chi\tanh\chi\cos^2\theta}{\bar{X}_2}\sigma_3\right)^2\Bigg]\,, \end{aligned} \tag{2.3}$$

with

$$\begin{aligned}
\Omega^2 &= \frac{\bar{X}_1^{1/2}\cosh\chi}{\rho}\,,\\
\bar{X}_1 &= \cos^2\theta+\rho^6\sin^2\theta\,,\\
\bar{X}_2 &= \mathrm{sech}\chi\cos^2\theta+\rho^6\cosh\chi\sin^2\theta\,.
\end{aligned}\tag{2.4}$$

The σ_i are the standard SU(2) left-invariant forms, the sum of the squares of which give the standard metric on a round three-sphere. They are normalised such that $d\sigma_i = \epsilon_{ijk}\sigma_j\wedge\sigma_k$. For future use, we shall denote the coordinates on the S^3 as $(\varphi_1,\varphi_2,\varphi_3)$.

The functions $\rho(r)\equiv e^{\alpha(r)}$ and $\chi(r)$ (to be discussed more in detail shortly) which appear in the ten-dimensional metric are the supergravity scalars coupling to certain operators in the dual gauge theory. There is a one-parameter family of solutions for them which gives therefore a family of supergravity solutions.

At $r\to\infty$, the UV, the various functions in the solution have the following asymptotic values:

$$\rho(r)\longrightarrow 1\,,\qquad \chi(r)\longrightarrow 0\,,\qquad A(r)\longrightarrow\frac{r}{L}\,.\tag{2.5}$$

This gives $AdS_5\times S^5$, and the cosmological constant is $\Lambda=-6/L^2$ where the normalisations are such that the gauge theory and string theory quantities are related to them as:

$$L=\alpha'^{1/2}\left(2g_{\mathrm{YM}}^2N\right)^{1/4};\qquad g_{\mathrm{YM}}^2=2\pi g_s\,.\tag{2.6}$$

This limit defines the SO(6) symmetric critical point of the $\mathcal{N}=8$ supergravity scalar potential where all of the 42 scalars vanish. At the end of the flow, in the IR $r\to-\infty$, the functions asymptote to the values:

$$\chi(r)\longrightarrow\frac{1}{2}\log 3\,,\qquad \alpha(r)\equiv\log\rho\longrightarrow\frac{1}{6}\ln 2\,,\qquad A(r)\longrightarrow\frac{2^{5/3}}{3}\frac{r}{L}\,.\tag{2.7}$$

which are the values defining another, SU(2) × U(1) symmetric, critical point of the scalar potential [6]. It preserves only $\mathcal{N}=2$ supersymmetry of the maximal $\mathcal{N}=8$ for five-dimensional supergravity.

It is easily seen that the non-trivial radial dependences of $\rho(r)$ and $\chi(r)$ deform the metric of the supergravity solution from $AdS_5\times S^5$ at $r=+\infty$ where there is an obvious SO(6) symmetry (the round S^5 is restored), to a spacetime which only has an SU(2) × U(1) symmetry, which is manifest in the metric of eq. (2.3).[4]

The fields Φ and $C_{(0)}$, the ten-dimensional dilaton and R-R scalar, are gathered into a complex scalar field $\lambda=C_{(0)}+ie^{-\Phi}$ on which $\mathrm{SL}(2,\mathbb{Z})$ has a natural action. This $\mathrm{SL}(2,\mathbb{Z})$ is the duality symmetry of the gauge theory in the UV (the dilaton is

[4]The SU(2) is the left-invariance of the σ_i and the U(1) rotates σ_1 into σ_2. Actually the metric has an extra U(1) symmetry, as $\partial/\partial\phi$ is also a Killing vector, but this is not a symmetry of the other fields in the full solution.

related to the gauge coupling $g_{\rm YM}$, and the R-R scalar to the Θ-angle), and an action of it will be inherited by the gauge theory in the IR. It was noticed in ref. [20] that λ is constant all along the flow. This is the most obvious sign that the geometry is going to be comparatively simple along the flow.

The non-zero parts of the two-form potential, $C_{(2)}$, and the NS-NS two-form potential $B_{(2)}$ are also presented in ref. [20], and they are non trivial. However, it turns out that the probe brane we study is aligned in such a way that their pull-back to the world-volume is exactly zero, so we do not exhibit them here.

We will need, however, the explicit form for the R-R four form potential $C_{(4)}$, to which the D3-brane naturally couples. The derivatives of this field, which appear in the field strength, are presented in ref. [20], and checks are made there on the mixed second derivatives in order to ensure consistency. However, after some algebra, it is possible to integrate the equations to yield a closed form for the potential, the relevant part of which which we write as:[5]

$$C_{(4)} = -\frac{4}{g_s} w(r,\theta)\, dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3\,, \tag{2.8}$$

where

$$w(r,\theta) = \frac{e^{4A}}{8\rho^2}\Big[\rho^6 \sin^2\theta(\cosh(2\chi)-3) - \cos^2\theta(1+\cosh(2\chi))\Big]\,. \tag{2.9}$$

The radial dependences of the functions $\rho(r), \chi(r)$, and $A(r)$, which appear in the ten-dimensional solution, were found to be governed by the reduction of the five-dimensional supergravity equations of motion to the following (recall that $\rho \equiv e^{\alpha}$):

$$\begin{aligned}
\frac{d\rho}{dr} &= \frac{1}{6L}\left(\frac{\rho^6(\cosh(2\chi)-3)+2\cosh^2\chi}{\rho}\right),\\
\frac{d\chi}{dr} &= \frac{1}{2L}\left(\frac{(\rho^6-2)\sinh(2\chi)}{\rho^2}\right),\\
\frac{dA}{dr} &= -\frac{1}{6L\rho^2}\big(\cosh(2\chi)(\rho^6-2)-(3\rho^6+2)\big)\,.
\end{aligned} \tag{2.10}$$

There is no known exact solution for these particular equations, but much can be deduced about the structure of the solution by resorting to numerical methods, as presented in ref. [10]. It should be noted that it is possible to extract the asymptotic UV ($r \to +\infty$) behaviour of the fields $\chi(r)$ and $\alpha(r) = \log(\rho(r))$ is given by:

$$\chi(r) \longrightarrow a_0 e^{-r/L} + \cdots\,; \qquad \alpha(r) \longrightarrow \frac{2}{3}a_0^2\frac{r}{L}e^{-2r/L} + \frac{a_1}{\sqrt{6}}e^{-2r/L} + \cdots\,. \tag{2.11}$$

Crucially, the values of the constant [10]

$$\hat{a} = \frac{a_1}{a_0^2} + \sqrt{\frac{8}{3}}\log a_0 \tag{2.12}$$

[5]By "relevant", we mean the part which gets pulled back to the D-brane aligned parallel to the (x^0, x^1, x^2, x^3) directions.

characterise a family of different solutions for $(\rho(r), \chi(r), A(r))$ representing different flows to the gauge theory in the IR. Meanwhile, in the IR ($r \to -\infty$) the asymptotic behaviour is:

$$\chi(r) \longrightarrow \frac{1}{2}\log 3 - b_0 e^{\lambda r/L} + \cdots; \qquad \alpha(r) \longrightarrow \frac{1}{6}\log 2 - \frac{\sqrt{7}-1}{6} b_0 e^{\lambda r/L} + \cdots,$$
$$\text{where} \qquad \lambda = \frac{2^{5/3}}{3}(\sqrt{7}-1)\,. \tag{2.13}$$

At this end of the flow, there is also a combination which characteristic of the flow, and this is $b_0 a_0^{\lambda}$. This has been pointed out in ref. [10] as characterising the width of the interpolating region.

The critical value [10] $\hat{a}_c \simeq -1.4694$ represents the particular flow which starts out at the $\mathcal{N}=4$ critical point and ends precisely on the $\mathcal{N}=1$ critical point. In ref. [15] it was proposed that the solutions with $\hat{a} > \hat{a}_c$ describe the gauge theory at different points on the Coulomb branch of moduli space. This fits with the fact that the behaviour of χ is, according to the dictionary [2, 3], characteristic of an operator of dimension three representing mass term (controlled by a_0), while that of α represents a *mixture* of both a dimension two mass operator (again through a_0) and a vacuum expectation value (vev) of an operator of mass two (through a_1). The combination $\hat{a}_c$ then, is pure mass and no vev, while other values are a mixture of both. The vev is that of a combination of massless fields which take us out onto the Coulomb branch. We shall discuss this more in section 3.

For the flows with the $\hat{a} < \hat{a}_c$, the five-dimensional supergravity potential is no longer bounded above by the asymptotic UV value and ref. [15] suggests that this makes them physically unacceptable. They correspond to attempting to give a positive vev to the massive field.

3. The gauge theory

The $\mathcal{N}=4$ supersymmetric Yang-Mills theory's gauge multiplet has bosonic fields (A_μ, X_i), $i = 1, \ldots, 6$, where the scalars X_i transform as a vector of the SO(6) R-symmetry, and fermions λ_i, $i = 1, \ldots, 4$ which transform as the **4** of the SU(4) covering group of SO(6).

In $\mathcal{N}=1$ language, there is a vector supermultiplet (A_μ, λ_4), and three chiral multiplets made of a fermion and a complex scalar ($k = 1, 2, 3$):

$$\Phi_k \equiv (\lambda_k, \phi_k = X_{2k-1} + iX_{2k})\,, \tag{3.1}$$

and they have a superpotential

$$W = h\,\mathrm{Tr}(\Phi_3[\Phi_1, \Phi_2])\,, \tag{3.2}$$

where h is related to $g_{\rm YM}$ in a specific way consistent with superconformal symmetry. We give a mass to Φ_3,

$$L_{\rm ft} \longrightarrow L_{\rm ft} + \int d^2\theta \, \frac{1}{2} m \Phi_3^2 + {\rm h.c.}\,, \tag{3.3}$$

and then flow from the $\mathcal{N} = 4$ gauge theory to the resulting $\mathcal{N} = 1$ theory. This theory has "matter" multiplets in two "flavours", Φ_1 and Φ_2, transforming in the adjoint of SU(N).

The SU(4) $\simeq$ SO(6) R-symmetry of the $\mathcal{N} = 4$ gauge theory is broken to $\mathrm{SU}(2)_F \times \mathrm{U}(1)_R$, the latter being the R-symmetry of the $\mathcal{N} = 1$ theory, and the former a flavour symmetry under which the matter multiplet forms a doublet.

So we switch on this small but relevant mass perturbation in the UV (and possibly a vev of some of the massless fields too) and flow to the IR. This maps to turning on certain scalar fields in the supergravity, whose values asymptote to zero. As one falls well below the scale of the mass m — going to the IR — these operators become more relevant. In the supergravity solution, this corresponds to the scalars being close to zero in the UV ($r \to +\infty$), developing a non-trivial profile as a function of r, becoming more significantly different from zero as one goes deeper into the IR, $r \to -\infty$. This is precisely what is captured in eqs. (2.11) and (2.13). Finally, the supergravity equations of motion require that there be a non-trivial back-reaction on the geometry, which deforms the spacetime metric in a way given by $A(r)$, etc. in section 2.

Specifically, we must consider a combination of the operators [10, 15]:

$$\begin{aligned} \alpha: \qquad & \sum_{i=1}^{4} {\rm Tr}(X_i X_i) - 2\sum_{i=5}^{6} {\rm Tr}(X_i X_i)\,, \\ \chi: \qquad & {\rm Tr}(\lambda_3\lambda_3 + \phi_1[\phi_2,\phi_3]) + {\rm h.c.}\,, \end{aligned} \tag{3.4}$$

and we have listed the corresponding scalar fields on the left. The specific nature of the terms is due to operator mixing and the manner in which they combine to give the pure mass deformation is discussed nicely in the review of ref. [34].

In fact, we can legitimately integrate out the massive scalar Φ_3 at a low enough scale, and this results in the quartic superpotential

$$W = \frac{h^2}{4m} {\rm Tr}([\Phi_1, \Phi_2]^2)\,, \tag{3.5}$$

which is in fact a marginal operator of the theory, defining a fixed line of theories generated by varying its coefficient [31].

The Coulomb branch moduli space of the $\mathcal{N} = 1$ SU(N) gauge theory is parameterised by the vevs of the complex adjoint scalars $\phi_{1,2}$ which set the potential ${\rm Tr}([\phi_1, \phi_2]^2)$ to zero. This generically breaks the theory to a product of U(1)'s.

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We shall only probe a four-dimensional subspace of the full moduli space here since our moduli space is the space of allowed zero-cost transverse movements of our single D3-brane probe. These directions are parameterised by the scalars (X_1, X_2, X_3, X_4), which make up the complex doublet (ϕ_1, ϕ_2). Moving in that hyperplane corresponds to the choice $\theta = 0$ in the coordinates presented earlier.

We shall find that the metric on this moduli space is very simple, and is topologically $\mathbb{R}^4$. It will naturally inherit the slightly squashed S^3 contained in the supergravity solution, and so only have the action of an $\mathrm{SU}(2)_F \times \mathrm{U}(1)_R$ global symmetry of the $\mathcal{N} = 1$ gauge theory.

4. Probing with a D3-brane

The uplifted geometry presented in ref. [20] and listed in section 2 is given in the Einstein frame. It is economical to write the D3-brane world-volume action in terms of this metric:

$$\begin{aligned} S = -\tau_3 \int_{\mathcal{M}_4} & d^4\xi \, \det{}^{1/2}[G_{ab} + e^{-\Phi/2}\mathcal{F}_{ab}] + \\ & + \mu_3 \int_{\mathcal{M}_4} \left(C_{(4)} + C_{(2)} \wedge \mathcal{F} + \frac{1}{2} C_{(0)} \, \mathcal{F} \wedge \mathcal{F} \right) , \end{aligned} \tag{4.1}$$

where $\mathcal{F}_{ab} = B_{ab} + 2\pi\alpha' F_{ab}$, and $\mathcal{M}_4$ is the world-volume of the D3-brane, with coordinates $\xi^0, \ldots, \xi^3$. As usual, the parameters μ_3 and τ_3 are the basic [35] R-R charge and tension of the D3-brane:

$$\mu_3 = \tau_3 g_s = (2\pi)^{-3}(\alpha')^{-2} . \tag{4.2}$$

Also, G_{ab} and B_{ab} are the pulls-back of the ten-dimensional metric (in Einstein frame) and the NS-NS two-form potential, respectively which is defined as e.g.:

$$G_{ab} = G_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} . \tag{4.3}$$

Working in static gauge, we partition the spacetime coordinates, x^μ, as follows: $x^i = \{x^0, x^1, x^2, x^3\}$, and $y^m = \{r, \theta, \phi, \varphi_1, \varphi_2, \varphi_3\}$. (The φ_i are angles on the deformed S^3 of section 2.) We choose static gauge as:

$$x^0 \equiv t = \xi^0 , \qquad x^i = \xi^i , \qquad y^m = y^m(t) . \tag{4.4}$$

Putting everything together, we get the following result for the effective lagrangian for the probe moving *slowly* in the transverse directions $y^m = (r, \phi, \theta, \varphi_1, \varphi_2$, $\varphi_3)$ (we restrict ourselves to considering $F_{ab} = 0$ here):

$$\mathcal{L} \equiv T - V = \frac{\tau_3}{2} \Omega^2 e^{2A} G_{mn} \dot{y}^m \dot{y}^n - \tau_3 \sin^2\theta e^{4A} \rho^4 (\cosh(2\chi) - 1) . \tag{4.5}$$

Where the G_{mn} refer to the Einstein frame metric components, and we have neglected terms higher than quadratic order in the velocities in constructing the kinetic term.

5. Coulomb branches

The logic of this whole approach is that the entire supergravity solution is made of coincident D3-branes, carrying an SU(N) gauge theory. The probe computation represents the pulling of a test brane out of the group, and exploring the background geometry and fields produced by all of the others, which is the solution (2.1), with accompanying fields. This breaks SU(N) $\rightarrow$ SU($N-1$) $\times$ U(1), and generically pulling them all apart would give U(1)$^{N-1}$, although we focus on the result for one probe at a time.

Let us orient ourselves by recalling the UV case. We can obtain this from our results by inserting the UV quantities given in eq. (2.5) into eq. (4.5). Our result *formally* (see below) gives the expected maximally supersymmetric case of vanishing potential, giving flatness in all six transverse directions to the brane.[6] The metric on this moduli space is simply the flat metric on $\mathbb{R}^6$:

$$ds^2_{\mathcal{M}_{\rm UV}} = \frac{1}{8\pi^2 g^2_{\rm YM}}\left[dv^2 + v^2 d\Omega_5^2\right] \qquad \text{with} \qquad v = \frac{L}{\alpha'}e^{r/L}\,, \tag{5.1}$$

where we have used the relations (2.6) and (4.2) and defined the energy scale v. Here, $d\Omega_5^2$ is the metric on a round S^5.

As stressed above, it is only formally correct to place those values into the probe lagrangian to get the answers above. We do *not* get that result as the smooth endpoint of the flow. A more careful limit of the potential should involve expanding the $\cosh(2\chi(r))$ term for large r, inserting the behaviour given in (2.11). This results in the behaviour

$$V(r) \sim e^{2r/L}\sin^2\theta\,, \tag{5.2}$$

showing that in fact only the directions specified by $\theta = 0$ are seen as flat directions in the UV limit, from the point of view of the effective physics on the brane probe. So in fact, we should replace $d\Omega_5^2$ in eq. (5.1) by $d\Omega_3^2$, the metric on a round S^3, since there are only four flat directions. This is in contrast to what one would deduce locally from the full supergravity solution (2.1), which in the UV limit goes smoothly to $AdS_5 \times S^5$.

The point is that the lagrangian on the probe at radius r yields a low energy effective action for the dual field theory below a cutoff defined roughly by v (or r). This effective action has knowledge of the entire theory below that scale (at least,[7]) especially the far IR, and so the probe cannot lose sight of the fact that the full

[6] See for example, the review of this sort of computation in ref. [23] for a fuller discussion.

[7] A *Wilsonian* [36] exact effective action would of course also know about the UV physics. The precise relation of the effective action on a D-brane probe to a Wilsonian action deserves more careful consideration. See e.g. refs. [37] for work on effective actions (Wilsonian or otherwise) in the AdS/CFT Correspondence. See also refs. [38] for the study of the Wilsonian exact effective action for gauge theory.

theory is actually not the $\mathcal{N}=4$ gauge theory, but the $\mathcal{N}=1$ theory. Note that this simple field theory fact has somewhat profound holographic implications, and we shall emphasise these points further in the discussion section.

So a general point on the flow has $\theta=0$ as the family of flat directions.[8] This moduli space is the Coulomb branch of the gauge theory anywhere along the flow. We see that we have movement on the (squashed) S^3, with coordinates $(\varphi_1, \varphi_2, \varphi_3)$, and the radial direction r. These give an $\mathbb{R}^4$, topologically, which is appropriate to the fact that we have two complex scalar fields in the adjoint, ϕ_1 and ϕ_2, whose vevs we can explore. The metric on this moduli space for arbitrary $(r, \varphi_1, \varphi_2, \varphi_3)$ is:

$$ds^2 = \frac{\tau_3}{2}\frac{\cosh^2\chi}{\rho^2}e^{2A}dr^2 + \frac{\tau_3}{2}L^2 e^{2A}\rho^2\left(\cosh^2\chi\,\sigma_3^2 + \sigma_1^2 + \sigma_2^2\right), \tag{5.3}$$

whose behaviour can be verified by a numerical study of the flow equations for the functions $(\chi(r), \rho(r), A(r))$.

We can study this metric in the IR limit $r \to -\infty$. Inserting the IR values of the functions (see eq. (2.7)), using the relations in eqs. (2.6) and (4.2), and defining:

$$u = \frac{\rho_0 L}{\alpha'}e^{r/\ell}, \qquad \ell = \frac{3}{2^{5/3}}L, \qquad \rho_0 \equiv \rho_{\rm IR} = 2^{1/6} \tag{5.4}$$

we get this extremely pleasing form for the metric:

$$ds^2_{\mathcal{M}_{\rm IR}} = \frac{1}{8\pi^2 g_{\rm YM}^2}\left[\frac{3}{4}du^2 + u^2\left(\frac{4}{3}\sigma_3^2 + \sigma_1^2 + \sigma_2^2\right)\right]. \tag{5.5}$$

There is a conical singularity at the origin, as a result of the $4/3$ coefficient of σ_3. The S^3's in constant r or u radial slices are squashed by the presence of $\cosh^2\chi$ (which is equal to $4/3$ in the IR) instead of unity. Such a deformation is to be expected, since the residual isometry is $\mathrm{SU}(2)_F \times \mathrm{U}(1)_R$, the global symmetry of the gauge theory. It would be interesting to find the gauge theory interpretation of this metric, including the singularity. The radial coordinate u that we have chosen in the IR is motivated by the choice (5.1) of radial coordinate v in the UV, where, because of the extra supersymmetry, the resulting v dependence corresponds to the vanishing of the β-function. It is interesting that we have a similar dependence here since under $u \to \lambda u$, the metric (5.5) recales by λ^2, for λ real. This is suggestive of conformal invariance, however we cannot directly conclude anything about the β-function since we only have $\mathcal{N}=1$ supersymmetry, which provides no relation between the kinetic term and the gauge coupling.

Further comparison to the gauge theory (yielding understanding of, e.g. the conical singularity) perhaps requires finding coordinates which are better adapted to the fixed point theory. This is the subject of ongoing research.

[8]The case $\rho=0$, which is $\alpha=-\infty$, lies outside the physically allowed values of the flow.

6. Discussion and holographic reflections

In summary, we have studied the results of probing a particular ten-dimensional type-IIB supergravity solution with a D3-brane. The solution has the dual interpretation as a renormalisation group flow from the maximally supersymmetric conformally invariant $\mathcal{N} = 4$ SU(N) gauge theory to a conformally invariant $\mathcal{N} = 1$ supersymmetric SU(N) gauge theory with two massless matter flavours in the adjoint. Both gauge theories are of course at strong 't Hooft coupling, $g^2_{\rm YM}N$, with large N and small $g_{\rm YM}$.

This sort of flow is expected to be simple, since the beginning and ends of the trajectory are conformal, and the spacetime is AdS (times a compact manifold) at both ends. This is reflected in the good behaviour of the probe problem everywhere along the flow, in necessary contrast to the results obtained for non-conformal flows in refs. [29, 30].

The piece of the Coulomb branch that the computation yields is consistent with expected properties of the gauge theory in the IR: there is a simple scaling property of the metric, with an isometry corresponding to the $\mathrm{SU}(2)_F \times \mathrm{U}(1)_R$ global symmetry. One can also solve for these flat directions anywhere along the flow, which is interesting.

Before closing we would like to highlight a remarkable holographic feature revealed by the probe computation. Recall that the supergravity goes back to the complete $AdS_5 \times S^5$ solution in the UV ($r \to +\infty$) limit, and forgets about the perturbation by the relevant operators to any desired accuracy if one goes to large enough r. This is not the case for the probe result. The effective lagrangian of the probe does not forget about the perturbation, since, as we pointed out in the previous section, an $\exp(2r/L)$ factor prevents us matching smoothly onto the result for a probe of pure AdS.

This is crucial and we emphasise it: *the ten-dimensional geometry of the flow solution becomes arbitrarily close to that of pure $AdS_5 \times S^5$ in the UV, but the physics of the D3-brane probing the flow geometry does not approach that seen by a D3-brane in pure $AdS_5 \times S^5$.*

This is counterintuitive from the point of view of the physics of local probes: one would expect that locally, for large positive r, a probe (such as one which knows only about the local metric) cannot tell that it is not in simple $AdS_5 \times S^5$, and the small deviations do not matter. This is not so for a D3-brane probe: it can tell the difference arbitrarily far into the UV ($r \to +\infty$), because of the $\exp(2r/L)$ factor discussed around eq. (5.2).[9]

From the point of view of the four-dimensional gauge theory physics, however, this is precisely the right expectation. The effective physics computed for the probe at any radius is an expression of the physics of the gauge theory at the effective

[9]See e.g. refs. [39] for some discussion of the issue of probes in AdS and holography.

scale defined by r (or u). So the probe's effective lagrangian should capture the deep infrared physics anywhere along the flow, and should in particular know that it is the $\mathcal{N}=1$ theory and not the $\mathcal{N}=4$. Therefore, no matter how far one runs into the UV (large r), the probe should be sensitive to the physics present in the IR (small r). A supergravity probe cannot manage this, but a D3-brane can, as shown in the simple, clear example of this paper.

Running the logic the other way, if one had two supergravity geometries which were similar in the IR but quite different in the UV, one would expect that the probe physics would be less sensitive to the differences in the UV, since those are short distance details that do not crucially affect the low energy effective action.

We remark that it is certainly worth studying these probe techniques further, applying them to other new (inevitably more complicated) ten-dimensional geometries which are appearing on the market regularly: they offer complementary insights into the physics that gauge/geometry dualities can teach us, and are a useful route by which we can bridge the gap between purely supergravity techniques and the full superstring technology needed to fully investigate these backgrounds.

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